Patterns Generation and Spatial Entropy in Multi-Dimensional Lattice Models

Song-Sun Lin,

Deaprtment of Applied Mathematics,

National Chiao Tung University

Joint works with

Dr. Jung-Chao Ban, NCTS, and Ms. Yin-Heng Lin, NCTU.

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Contents:

§1. Introduction

§§1.1 Motivations

(I) Lattice Dynamical System (LDS)

1. L. O. CHUA and L. YANG, Cellular neural networks: theory and applications, IEEE Trans. Circuits Systems, 35(1988), pp. 1257-1290.
2. L. O. CHUA, CNN: A paradigm for complexity. World Scientific Series on Nonlinear Science, Series A, 31. World Scientific, Singapore. (1998).
3. S. N. CHOW, J. MALLET-PARET and E. S. VAN VLECK, Pattern formation and spatial chaos in spatially discrete evolution equations, Random Comput. Dynam., 4(1996), pp. 109-178.

(II) 2D Patterns Generation & Spatial Entropy

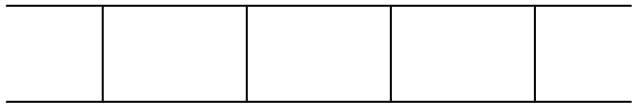
4. J. JUANG and S. S. LIN, Cellular neural networks: mosaic pattern and spatial

chaos, SIAM J. Appl. Math., 60(2000), pp.891-915.

5. C. H. HSU, J. JUANG, S. S. LIN, and W. W. LIN, Cellular neural networks: local patterns for general template, International J. of Bifurcation and Chaos, 10(2000), pp.1645-1659.
6. J. JUANG, S. S. LIN, W. W. LIN and S. F. SHIEH, The spatial entropy of two-dimensional subshifts of finite type, International J. of Bifurcation and Chaos, 10(2000), pp.2845-2852.
7. S. S. LIN and T. S. YANG, On the spatial entropy and patterns of two-dimensional cellular neural network, International J. of Bifurcation and Chaos, 12 (2002), 115-128.
8. J. C. BAN and S. S. LIN, Patterns generation and transition matrices in multi-dimensional lattice models. Discrete and Conti. Dyn. Sys., 2005.
9. J. C. Ban, S. S. Lin and Y. H. Lin, Patterns generation and spatial entropy in multi-dimensional lattice models, (preprint 2004).

§§1.2 1-D case

<u>1-D Lattice</u> $\mathbf{Z}^1$



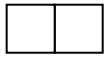
<u>Symbols</u> (colors, alphabets)

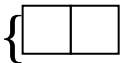
$S=\{0, \ldots, p\text{-}1\}$.

Two symbols $S=\{0,1\}$; two colors $S=\{$ □ , ■ $\}$.

<u>Local Admissible Conditions</u> (Local Interaction Property) The state at each lattice point only influenced by its finitely many neighborhood states.

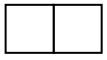
Example 1.1.

On $\mathbf{Z}_2 \equiv$  (Basic lattice)

$\Sigma_2 = \{$  ,  ,  ,  $\}$, the set of all local patterns defined on $\mathbf{Z}_2$.

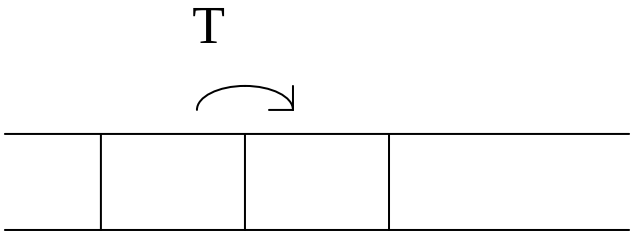
$B \subset \Sigma_2$: Basic (admissible) set, the set of all admissible local patterns.

Example 1.2.

B={  ,  ,  }; basic set.

Transition matrix : $T = T(B)$

$$\begin{array}{c|cc} & \square & \blacksquare \\ \hline \square & 1 & 1 \\ \blacksquare & 1 & 0 \end{array} .$$



Questions

$\Sigma_n(\mathrm{B})$: all admissible patterns on $\mathbf{Z}_n$ which can be generated by B.

(a) How to generate $\Sigma_n(B)$ from B ? $\Gamma_n(B) \equiv \#\Sigma_n(B) =$?

(b) Spatial Entropy : $h(B) \equiv \lim_{n\to\infty} \frac{\log \Gamma_n(B)}{n} =$?

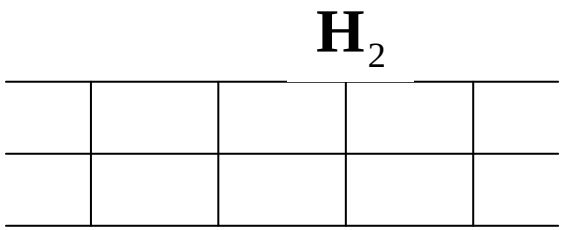
Answers

(a) $\Sigma_n(\mathrm{B}) = T^{n-1}$ and $\Gamma_n(B) = \left|\mathrm{T}^{n-1}\right| = \sum_{i=1}^{2}\sum_{j=1}^{2} (\mathrm{T}^{n-1})_{ij}$.

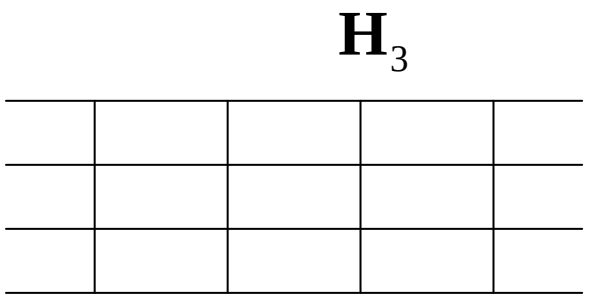
(b) $h(B) = \log \rho(T)$, where $\rho(T)$ is the maximum eigenvalue of $\mathrm{T} = \mathrm{T}(B)$.

§2 Two dimensional lattices

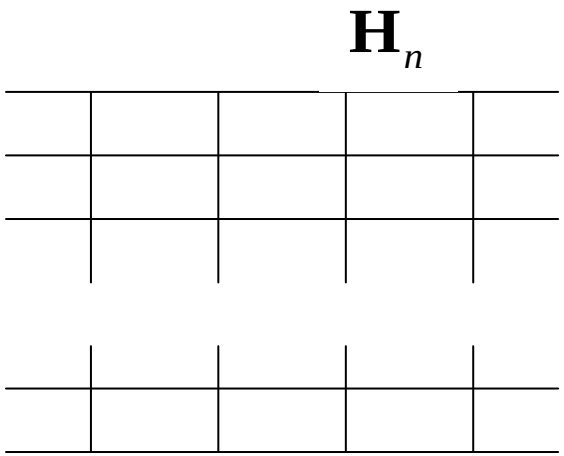
$\mathbf{H}_2$: transition matrix for $\mathbf{Z}_{\infty\times 2}$



$\mathbf{H}_3$: transition matrix for $\mathbf{Z}_{\infty\times 3}$



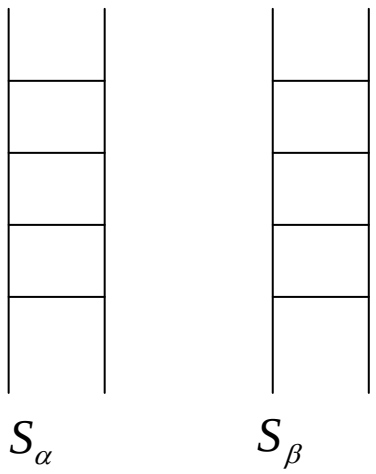
$\mathbf{H}_n$: transition matrix for $\mathbf{Z}_{\infty\times n}$



$\mathbf{H}_\infty$: transition matrix for $\mathbf{Z}_{\infty\times\infty}$

: $S^{\mathbf{Z}^1} \times S^{\mathbf{Z}^1} \to \{0,1\}$

$S^{\mathbf{Z}^1}$: The set of symbols (vertical strips) is uncountable.



$$\mathbf{H}_\infty(S_\alpha, S_\beta) = \begin{cases} 1 & \text{admissible,} \\ 0 & \text{forbidden.} \end{cases}$$

$\mathbf{H}_\infty$

Questions

(i) What is the relation between $\mathbf{H}_n$ & $\mathbf{H}_{n+1}$?

i.e. can we obtain a recursive formula for $\mathbf{H}_{n+1}$ in terms of $\mathbf{H}_n$, …, $\mathbf{H}_2$?

(ii) How to compute $\rho_n = \rho(\mathbf{H}_n)$ and the spatial entropy

$$h(\mathbf{H}_2) = \lim_{n\to\infty} \frac{\log \rho(\mathbf{H}_n)}{n} \quad ?$$

(iii) What is the relation between ρ_{n+1} and ρ_n , ρ_{n-1} , …, ρ_2 ?

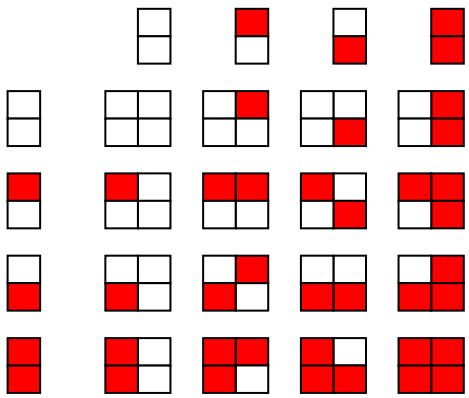
(iv) If $\rho_* = \lim_{n\to\infty} \rho_n^{1/n}$, $h = \log \rho_*$, any "limiting" equation satisfied by ρ_* ?

§§2.1 Ordering Matrices

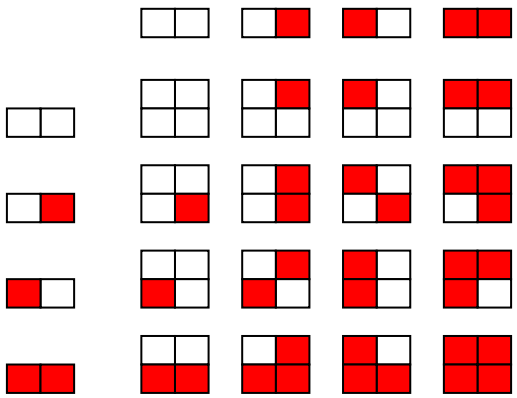
Ref [8]：(Ban - L.)

On $\mathbf{Z}_{2\times 2}$, $\Sigma_{2\times 2}$ can be arranged by

$\mathbf{X}_2 = [x_{i_1,i_2}] =$



$\mathbf{Y}_2 = [y_{j_1,j_2}] =$



Orderings of $\mathbf{Z}_{2\times 2}$ by

2	4
1	3

&

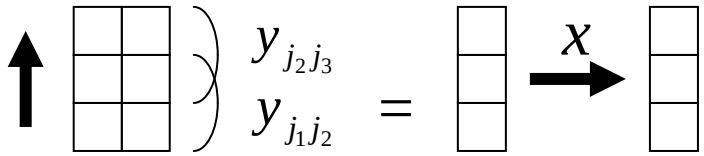
3	4
1	2

respectively.

Observations

$$\mathbf{X}_2 = \left[\begin{array}{cc|cc} y_{11} & y_{12} & y_{21} & y_{22} \\ y_{13} & y_{14} & y_{23} & y_{24} \\ \hline y_{31} & y_{32} & y_{41} & y_{42} \\ y_{33} & y_{34} & y_{43} & y_{44} \end{array}\right] \equiv \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \rightarrow & 2 \\ & \swarrow & \\ 3 & \rightarrow & 4 \end{bmatrix} \qquad \left[\begin{array}{c|c} 1 \rightarrow 2 & 1 \rightarrow 2 \\ \swarrow & \swarrow \\ 3 \rightarrow 4 & 3 \rightarrow 4 \\ \hline 1 \rightarrow 2 & 1 \rightarrow 2 \\ \swarrow & \swarrow \\ 3 \rightarrow 4 & 3 \rightarrow 4 \end{array}\right]$$

- For $\Sigma_{2\times 3}$, 

.
- $y_{j_1 j_2 j_3} \equiv y_{j_1 j_2} \hat{\oplus} y_{j_2 j_3}$.

§3 Transition Matrices

On $\mathbf{Z}_{2\times 2}$, given an admissible set $B \subset \Sigma_{2\times 2}$, define (horizontal) transition matrix $\mathbf{H}_2 \equiv \mathbf{H}_2(B) = [h_{j_1 j_2}]_{4\times 4}$, where $h_{j_1 j_2} \in \{0,1\}$ and $h_{j_1 j_2} = 1 \Leftrightarrow x_{j_1 j_2} \in B$.

$$\mathbf{H}_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} = \left[\begin{array}{cc|cc} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ \hline v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{array}\right] \equiv \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \quad \left(\equiv \begin{bmatrix} H_{2;1} & H_{2;2} \\ H_{2;3} & H_{2;4} \end{bmatrix} \right),$$

$$\mathbf{V}_2 = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} = \left[\begin{array}{cc|cc} h_{11} & h_{12} & h_{21} & h_{22} \\ h_{13} & h_{14} & h_{23} & h_{24} \\ \hline h_{31} & h_{32} & h_{41} & h_{42} \\ h_{33} & h_{34} & h_{43} & h_{44} \end{array}\right] \equiv \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} \quad \left(\equiv \begin{bmatrix} V_{2;1} & V_{2;2} \\ V_{2;3} & V_{2;4} \end{bmatrix} \right).$$

Denoted by $v_{j_1 j_2 j_3} \equiv v_{j_1 j_2} v_{j_2 j_3}$,

$$\mathbf{H}_3 = \left[\begin{array}{cc:cc|cc:cc} v_{111} & v_{112} & v_{121} & v_{122} & v_{211} & v_{212} & v_{221} & v_{222} \\ v_{113} & v_{114} & v_{123} & v_{124} & v_{213} & v_{214} & v_{223} & v_{224} \\ \hdashline v_{131} & v_{132} & v_{141} & v_{142} & v_{231} & v_{232} & v_{241} & v_{242} \\ v_{133} & v_{134} & v_{143} & v_{144} & v_{233} & v_{234} & v_{243} & v_{244} \\ \hline v_{311} & v_{312} & v_{321} & v_{322} & v_{411} & v_{412} & v_{421} & v_{422} \\ v_{313} & v_{314} & v_{323} & v_{324} & v_{413} & v_{414} & v_{423} & v_{424} \\ \hdashline v_{331} & v_{332} & v_{341} & v_{342} & v_{431} & v_{432} & v_{441} & v_{442} \\ v_{333} & v_{334} & v_{343} & v_{344} & v_{433} & v_{434} & v_{443} & v_{444} \end{array}\right] = \left[\begin{array}{cc|cc} v_{11}H_1 & v_{12}H_2 & v_{21}H_1 & v_{22}H_2 \\ v_{13}H_3 & v_{14}H_4 & v_{23}H_3 & v_{24}H_4 \\ \hline v_{31}H_1 & v_{32}H_2 & v_{41}H_1 & v_{42}H_2 \\ v_{33}H_3 & v_{34}H_4 & v_{43}H_3 & v_{44}H_4 \end{array}\right] \equiv \begin{bmatrix} H_{3;1} & H_{3;2} \\ H_{3;3} & H_{3;4} \end{bmatrix}$$

<u>Theorem 3.1.</u>(Ban-L.) Let $\mathbf{H}_2$ be a transition matrix, write

$$\mathbf{H}_n = \begin{bmatrix} H_{n;\ 1} & H_n \\ H_{n;\ 3} & H_n \end{bmatrix}; \text{ and } \mathbf{H}_{n+1} = \begin{bmatrix} H_{n+1;1} & H_{n+1;2} \\ H_{n+1;3} & H_{n+1;4} \end{bmatrix}.$$

Then

$$H_{n+1;k} = \begin{bmatrix} v_{k1} H_{n\ ;\ 1} & v_k\ H_{n2} \\ v_{k3} H_{n;3} & v_{k4} H_{n;4} \end{bmatrix}. \tag{3.1}$$

Furthermore,

$$\mathbf{H}_n = (\mathbf{H}_{n-1})_{2^{n-1}\times 2^{n-1}} \circ \left(E_{2^{n-2}} \otimes \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \right) \equiv \otimes \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}^{n-2} \tag{3.2}$$

, where $\mathbf{E}_{2^k}$ is the $2^k \times 2^k$ full matrix, i.e., all entries are 1. ■

<u>Example 3.2.</u> (Golden - Mean): $\mathbf{H} = \mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\mathbf{H}_2 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]_{4\times 4} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow \quad \mathbf{H}_3 = \begin{bmatrix} A & B & A & 0 \\ C & 0 & C & 0 \\ A & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{8\times 8}$$

Remark 3.3. When $\mathbf{H}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$, $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$

and $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$, $\rho(\mathbf{H}_n)$ can be found explicitly and limiting

equation for $\xi = \lim_{n\to\infty} \rho(\mathbf{H}_n)^{\frac{1}{n}} = \exp(h(\mathbf{H}_2))$ can also be found：

$$Q(\xi) = \begin{cases} 4\xi^2(\xi - a) + (\gamma^2 - 4\xi)(\xi - a)^2 - \gamma^2\xi^2 - 2\gamma(2b - a\gamma)\xi \\ \qquad -(2b - a\gamma)^2 \quad if \quad a_2 a_3 = 1, \\ \xi^3 - a\xi^2 - \delta\xi + a\delta - b \quad if \quad a_2 a_3 = 0 \quad \& \quad a_2 b_3 + a_3 b_2 = 1, \\ \qquad where \quad \gamma = b_2 + b_3 \quad and \quad \delta = b_2 b_3. \end{cases}$$

§4. Reduction Operators for $\mathbf{H}_n^m$ in n

$\mathbf{H}_n^m$: all admissible patterns on $\mathbf{Z}_{(m+1)\times n}$, and

$$h(\mathbf{H}_2) \equiv \lim_{m,n\to\infty} \frac{\log|\mathbf{H}_n^m|}{mn}$$

$$= \lim_{n\to\infty} \frac{1}{n} \left\{ \lim_{m\to\infty} \frac{\log|\mathbf{H}_n^m|}{m} \right\} = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbf{H}_n), \tag{4.1}$$

or

$$= \lim_{m\to\infty} \frac{1}{m} \left\{ \lim_{n\to\infty} \frac{\log|\mathbf{H}_n^m|}{n} \right\} \tag{4.2}$$

$$= \lim_{m\to\infty} \frac{1}{m} \left\{ \limsup_{n\to\infty} \frac{\log tr(\mathbf{H}_n^m)}{n} \right\}. \tag{4.3}$$

To use (4.2) or (4.3) to compute spatial entropy $h(\mathbf{H}_2)$, need to answer

<u>Questions</u> Fixed $m \geq 2$, for any $n \geq 2$

(i) Find recursive formulas from $\mathbf{H}_n^m$ to $\mathbf{H}_{n+1}^m$.

(ii) Find recursive formulas from $tr(\mathbf{H}_n^m)$ to $tr(\mathbf{H}_{n+1}^m)$.

<u>Notations</u>

$$\mathbf{H}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix} \overset{or}{=} \begin{bmatrix} H_{n;11} & H_{n;12} \\ H_{n;21} & H_{n;22} \end{bmatrix} \tag{4.4}$$

for extension to $\mathbf{H}_{n+1}$ for matrix multiplication

When $m = 2$,

$$\mathbf{H}_n^2 = \begin{bmatrix} H_{n;11}^2 + H_{n;12}H_{n;21} & H_{n;11}H_{n;12} + H_{n;12}H_{n;22} \\ H_{n;21}H_{n;11} + H_{n;22}H_{n;21} & H_{n;21}H_{n;12} + H_{n;22}^2 \end{bmatrix}.$$

Denote by

$$\mathbf{X}_{2,n} \equiv \begin{bmatrix} X_{2,n;1} & X_{2,n;2} \\ X_{2,n;3} & X_{2,n;4} \end{bmatrix} \text{ and } X_{2,n;1} = \begin{bmatrix} H_{n;11}^2 \\ H_{n;12}H_{n;21} \end{bmatrix}, \quad X_{2,n;2} = \begin{bmatrix} H_{n;11}H_{n;12} \\ H_{n;12}H_{n;22} \end{bmatrix},$$

$$X_{2,n;3} = \begin{bmatrix} H_{n;21}H_{n;11} \\ H_{n;22}H_{n;21} \end{bmatrix}, \quad X_{2,n;4} = \begin{bmatrix} H_{n;21}H_{n;12} \\ H_{n;22}^2 \end{bmatrix}.$$

Similarly, for any $m \geq 3$, denote

$$\mathbf{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} \\ X_{m,n;3} & X_{m,n;4} \end{bmatrix},$$

which represents all "elementary patterns" in $\mathbf{H}_n^m$. (3.1) $\Rightarrow$

$$\mathbf{X}_{m,n+1} = \begin{bmatrix} X_{m,n+1;1} & X_{m,n+1;2} \\ X_{m,n+1;3} & X_{m,n+1;4} \end{bmatrix} \text{ and } X_{m,n+1;i} = \begin{bmatrix} X_{m,n+1;i,1} & X_{m,n+1;i,2} \\ X_{m,n+1;i,3} & X_{m,n+1;i,4} \end{bmatrix}. \quad (4.5)$$

where $X_{m,n+1;i,j}$ is consist of products of $v_{kl} H_{n;l}$.

Then a recursive relation from $\mathbf{X}_{m,n}$ to $\mathbf{X}_{m,n+1}$ (or between $X_{m.n+1;i,j}$ and $X_{m.n;,j}$) are given as follows:

Theorem 4.1.

$X_{2,n+1;i,j} = S_{2;ij} X_{2,n;j}$, where

$$\mathbf{R}_2 = \begin{bmatrix} R_{2;11} & R_{2;12} & R_{2;13} & R_{2;14} \\ R_{2;21} & R_{2;22} & R_{2;23} & R_{2;24} \\ R_{2;31} & R_{2;32} & R_{2;33} & R_{2;34} \\ R_{2;41} & R_{2;42} & R_{2;43} & R_{2;44} \end{bmatrix} = \left[\begin{array}{cc|cc} S_{2;11} & S_{2;12} & S_{2;21} & S_{2;22} \\ S_{2;13} & S_{3;14} & S_{2;23} & S_{2;24} \\ \hline S_{2;31} & S_{2;32} & S_{2;41} & S_{2;42} \\ S_{2;33} & S_{2;34} & S_{2;43} & S_{2;44} \end{array}\right] \tag{4.6}$$

and

$$R_{2;ij} = \begin{bmatrix} h_{i1} & h_{i2} \\ h_{i3} & h_{i4} \end{bmatrix} \circ \begin{bmatrix} h_{1j} & h_{2j} \\ h_{3j} & h_{4j} \end{bmatrix}. \tag{4.7}$$

Furthermore, for $m \geq 3$, denote by

$$R_{m;ij} = \left\{ \begin{bmatrix} h_{i1} & h_{i2} \\ h_{i3} & h_{i4} \end{bmatrix} \circ \left(\hat{\otimes} \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}^{m-2} \right)_{2\times 2} \right\}_{2^{m-1}\times 2^{m-1}} \circ \left\{ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} h_{1j} & h_{2j} \\ h_{3j} & h_{4j} \end{bmatrix} \right\}_{2^{m-1}\times 2^{m-1}} \quad (4.8)$$

and

$$\mathbf{R}_m = \begin{bmatrix} R_{m;11} & R_{m;12} & R_{m;13} & R_{m;14} \\ R_{m;21} & R_{m;22} & R_{m;23} & R_{m;24} \\ R_{m;31} & R_{m;32} & R_{m;33} & R_{m;34} \\ R_{m;41} & R_{m;42} & R_{m;43} & R_{m;44} \end{bmatrix} = \left[\begin{array}{cc|cc} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ \hline S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{array} \right],$$

then,

$$X_{m,n+1;i,j} = S_{m;ij} X_{m,n;j} \,. \quad (4.9) \quad \blacksquare$$

<u>Theorem 4.2.</u> (Lower-bound of entropy)

For any $m \geq 2$, and $K \geq 1$, with $\beta_j \in \{1,4\},\ 1 \leq j \leq K$. Then

$$h(\mathbf{H}_2) \geq \frac{1}{mK} \log \rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1}). \quad (4.10)$$

<u>Example 4.3.</u> (Golden - Mean)

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_{m;11} = \mathbf{H}_{m-1}$$

$$\Rightarrow \rho(\mathbf{H}_m)^{\frac{1}{m+1}} \leq \exp(h(\mathbf{H}_2)) \leq \rho(\mathbf{H}_m)^{\frac{1}{m}}, \text{ for } m \geq 2.$$

Example 4.4.

$$\mathbf{H}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad R_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; R_{2;22} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; R_{2;33} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; R_{2;44} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$S_{2;14} S_{2;41} = R_{2;22} R_{2;33} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Rightarrow h(\mathbf{H}_2) \geq \frac{\log 2}{4} \quad .$$

§5. Trace Operators for $\mathbf{T}_m$

(4,3) can be used for computing and finding upper bounds of spatial entropy $h(\mathbf{H}_2)$.

<u>Theorem 5.1.</u> If $\mathbf{H}_2$ is symmetry, then $\mathbf{H}_n$ is symmetry for all $n \geq 3$, and

$$\rho(\mathbf{H}_n)^2 \leq tr(\mathbf{H}_n^2). \tag{5.1}$$

<u>Trace operator</u> $\mathbf{T}_m \equiv \begin{bmatrix} R_{m,11} & R_{m,22} \\ R_{m,33} & R_{m,44} \end{bmatrix}$.

Theorem 5.2.

For any $m \geq 2$, and $n \geq 2$, $tr(\mathbf{H}_n^m) = |\mathbf{T}_m^{n-1}|$. (5.2)

Furthermore,

$$\limsup_{n\to\infty} \frac{\log tr(\mathbf{H}_n^m)}{n} = \rho(\mathbf{T}_m) \tag{5.3}$$

and

$$h(\mathbf{H}_2) = \limsup_{m\to\infty} \frac{\log \rho(\mathbf{T}_m)}{m}. \tag{5.4}$$

When $\mathbf{H}_2$ is symmetric, for any $m \geq 1$, $h(\mathbf{H}_2) \leq \dfrac{\log \rho(\mathbf{T}_{2m})}{2m}$.

§§5.1. Simplified trace Operator $\boldsymbol{J}_m$

Using $tr(AB) = tr(BA)$, the $2^m \times 2^m$ trace operator $\mathbf{T}_m$ can be reduced to a $m^* \times m^*$ trace operator $\boldsymbol{J}_m$, where

$$m^* = [\frac{m}{2}][\frac{m+1}{2}] + 2 . \tag{5.5}$$

For each $m \geq 2$, let $0 \leq l_{1,}\ \ l_{2,}\ \ l_3$ and $l_1 + 2l_2 + l_3 = m$.

Ordering $H_{n;11}^{l_1}(H_{n;12}H_{n;21})^{l_2} H_{n;22}^{l_3}$ by the anti-lexicographic order in (l_1, l_2, l_3).

Denote by

$$t_{m,n;l_1,l_2,l_3} = tr(H^{l_1}_{n;11} H^{l_2}_{n;12} H^{l_2}_{n;21} H^{l_3}_{n;22}),$$

and

$$t_{m,n} = (t_{m,n;l_1,l_2,l_3})^t$$, a m^* - vector. Then we have

<u>Theorem 5.3.</u> For any $m \geq 2,$ there is a (simplified) trace operator $\boldsymbol{J}_m$ such that for any $n \geq 2$

$$t_{m,n+1} = \boldsymbol{J}_m t_{m,n} \tag{5.6}$$

and

$$\rho(\boldsymbol{J}_m) = \rho(\mathbf{T}_m). \tag{5.7}$$

Example 5.4.

$$\boldsymbol{J}_2 = \begin{bmatrix} h_{11}^2 & h_{12}h_{21} & h_{22}^2 \\ 2h_{13}h_{31} & h_{14}h_{41} + h_{23}h_{32} & 2h_{24}h_{42} \\ h_{33}^2 & h_{34}h_{43} & h_{44}^2 \end{bmatrix},$$

$$\boldsymbol{J}_3 = \begin{bmatrix} h_{11}^3 & h_{11}h_{12}h_{21} & h_{12}h_{22}h_{21} & h_{22}^3 \\ 3h_{11}h_{13}h_{31} & h_{11}h_{14}h_{41} + h_{12}h_{23}h_{31} + h_{21}h_{13}h_{32} & h_{12}h_{24}h_{41} + h_{21}h_{14}h_{42} + h_{22}h_{23}h_{32} & 3h_{22}h_{24}h_{42} \\ 3h_{13}h_{33}h_{31} & h_{13}h_{34}h_{41} + h_{14}h_{43}h_{31} + h_{23}h_{33}h_{32} & h_{14}h_{44}h_{41} + h_{23}h_{34}h_{42} + h_{24}h_{43}h_{32} & 3h_{24}h_{42}h_{44} \\ h_{33}^3 & h_{33}h_{34}h_{43} & h_{34}h_{44}h_{43} & h_{44}^3 \end{bmatrix}.$$

<u>Example 5.5.</u> (Simplified Golden - Mean)

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{H}_{n+1} = \begin{bmatrix} \mathbf{H}_n & \hat{\mathbf{H}}_{n-1} \\ \hat{\mathbf{H}}_{n-1} & 0 \end{bmatrix} \text{ where } \hat{\mathbf{H}}_{n-1} = \begin{bmatrix} \mathbf{H}_{n-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \quad \boldsymbol{J}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad \boldsymbol{J}_4 = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{J}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 3 & 1 & 0 \\ 9 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{J}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 8 & 5 & 3 & 1 & 0 \\ 20 & 6 & 1 & 0 & 0 \\ 16 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

m	$\rho(\mathbf{H}_m)^{\frac{1}{m+1}}$	$\rho(\boldsymbol{J}_m)^{\frac{1}{m}}$	$\rho(\mathbf{H}_m)^{\frac{1}{m}}$
2	1.25992	<u>1.41421</u>	1.41421
3	1.29514	1.32054	1.41174
4	1.29841	<u>1.35019</u>	1.38601
5	1.300843	1.33977	1.3711
6	1.31204	<u>1.34688</u>	1.37279
7	1.31639	1.36987	1.36911
8	1.31902	<u>1.33328</u>	1.36547
9	1.32149		1.36306

§6. Summary

1. Higher order transition matrices $\mathbf{H}_n$, $n \geq 3$, can be recursively derived from $\mathbf{H}_2$. (Ref. 8)
2. Lower-bound of entropy can be found by introducing reduction operator $\mathbf{R}_m$, $m \geq 2$. A powerful method to verify the positivity of entropy. (Ref. 9)
3. Trace (and simplified trace) operator $\mathbf{T}_m$ (and $\boldsymbol{J}_m$) have been introduced to compute and give a upper-bound of entropy. (Ref. 9)